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Equivalence of deformed fermionic algebras

Dennis Bonatsos† and C Daskaloyannis‡

† Institute of Nuclear Physics, NCSR 'Demokritos', GR-15310 Aghia Paraskevi, Attiki, Greece

‡ Theoretical Physics Department, University of Thessaloniki, GR-54006 Thessaloniki, Greece

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Abstract. Generalized deformations of the fermionic algebra are studied. The polynomial representations of these algebras are constructed. All the deformation schemes can be realized by the same polynomial basis (using the Bargmann representation), thus proving that all deformed fermionic algebras are isomorphically equivalent to the non-deformed fermionic algebra.

1. Introduction

The quantum deformation (or q-deformation) of classical algebras [1–3] as $\mathrm{SU}_q(2)$ emerged as a mathematical tool from the study of the quantum inverse problem, the Yang–Baxter equation and the conformal field theories. (A collection of the original papers can be found in [4].) The classical (non-deformed) algebras can be constructed using the harmonic oscillator algebra $\{a, a^+, N\}$ as the basic underlying structure. Biedenharn [5] and Macfarlane [6] constructed the q-deformed oscillator operators appropriate for deriving the generators of the q-deformed $\mathrm{SU}_q(2)$. Many authors have also studied the q-deformed counterparts of specific classical [7–9] and exceptional [10–12] algebras. For the realization of q-deformed superalgebras, however, in addition to the q-deformed boson oscillators one has to introduce q-deformed fermion oscillators. q-deformed fermions have, in fact, been introduced by several authors [11–16] and have, furthermore, been employed in obtaining oscillator realizations of q-deformed superalgebras [17–20].

The q-deformation scheme for bosons [5,6] is based on the assumption that the oscillator algebra basis $\{1, a, a^+, N\}$ satisfies the relations

$$aa^+ - qa^+a = q^{-N} \qquad [a, N] = a \qquad [a^+, N] = -a^+. \tag{1}$$

These commutation relations define a special deformation scheme. In the literature, however, one can find several different deformation schemes, such as, for example, the Q-deformed oscillator of Arik–Coon [21] and Kuryshkin [22], the two-parameter deformed oscillator [23–25], the parabosonic and parafermionic oscillators [26] and their q-deformations [27,28]. The common feature of all these deformations is their structural similarity. In all cases an appropriate Fock representation exists, so that all the notions derived from the oscillator algebra (such as oscillator realizations of the classical groups, coherent and squeezed states) can also be defined for all these algebras.

It is thus natural to assume that these deformed algebras are partial realizations of a generalized deformed algebra. There have indeed been several attempts towards describing these algebras in a unified framework. We mention here the Odaka–Kishi–Kamefuchi unification method [28], the Beckers–Debergh method [29], the generalized deformed oscillator [30–32] (which has found application in the description of pairing correlations in a single-j shell [33, 34], as well as in the description of vibrational spectra of diatomic molecules [35]), the bosonization method [36, 37] and the generalized Q-deformed oscillator [38].

In this paper various q-deformations of the fermionic algebra will be studied. A polynomial basis of the fermionic algebra will be subsequently constructed, which is a Bargmann realization of the algebra. It will be proven that all the deformed fermionic algebras possess the same polynomial realization if the creation and annihilation fermionic operators satisfy the relations $a^2 = (a^+)^2 = 0$. Therefore all the deformed fermionic schemes are isomorphically equivalent. As a result, although several different types of q-deformed boson exist, sometimes having different physical behaviours (see [33] for an example), only one type of fermion exists, which is equivalent to the usual fermion.

In section 2 of this paper a simplified version of the generalized deformation scheme of [30] is given. The deformed fermionic algebra is studied for the various deformation schemes in section 3, while in section 4 the equivalence of all these schemes is proven through the construction of a polynomial representation. Finally, section 5 contains a discussion of the present results and plans for further work.

2. The generalized deformed oscillator algebra

The generalized deformed oscillator was introduced in [30]. Here we give a new simplified version of this method. A deformed oscillator is defined by the algebra generated by the operators $\{1, a, a^+, N\}$ and the *structure function* $F(x)$, satisfying the relations:

$$[a, N] = a \qquad [a^+, N] = -a^+ \tag{2}$$

and

$$a^+ a = F(N) = [N] \qquad a a^+ = F(N+1) = [N+1] \tag{3}$$

where $F(x)$ is a positive analytic function with $F(0) = 0$ and the operator N is a *number* operator. From equation (3) the following commutation and anticommutation relations are obviously satisfied:

$$[a, a^+] = [N+1] - [N] \qquad \{a, a^+\} = [N+1] + [N]. \tag{4}$$

The *structure function* $F(x)$ is characteristic of the deformation scheme. In table 1 we give the structure functions corresponding to the different deformed oscillators

Table 1. Structure functions of special deformation schemes.

	$F(x)$	Reference
i	x	Harmonic oscillator
ii	$\dfrac{q^x - q^{-x}}{q - q^{-1}}$	q-deformed harmonic oscillator [5,6]
iii	$\dfrac{q^x -}{q - 1}$	Arik–Coon, Kuryshkin or Q-deformed oscillator [21,22]
iv	$\dfrac{q^x - p^{-x}}{q - p^{-1}}$	Two-parameter deformed oscillator [23–25]
v	$x(p+1-x)$	Parafermionic oscillator [26]
vi	$\dfrac{\sinh(\tau x)\sinh(\tau(p+1-x))}{\sinh^2(\tau)}$	q-deformed parafermionic oscillator [27,28]
vii	x^n	[30]
viii	$\dfrac{\mathrm{sn}(\tau x)}{\mathrm{sn}(\tau)}$	[30]

of [5, 6, 21–28, 30]. All these deformed oscillators are described by the same unified theory presented above.

If $h(z)$ is an entire function, then the following properties are true:

$$h(N)(a^+)^m = (a^+)^m h(N+m) \tag{5}$$

$$h(N+m)(a)^m = (a)^m h(N). \tag{6}$$

The generalized deformed algebras possess a Fock space of eigenvectors of the number operator $N = F^{-1}(a^+a)$, where the function F^{-1} must be analytic in the vicinity of zero and simultaneously invertible. These eigenvectors are generated by the formula:

$$|n\rangle = \frac{1}{\sqrt{[n]!}}(a^+)^n|0\rangle \tag{7}$$

where

$$[n]! = \prod_{k=1}^{n}[k] = \prod_{k=1}^{n} F(k). \tag{8}$$

The generators a^+ and a are the creation and annihilation operators of this deformed oscillator algebra:

$$a|n\rangle = \sqrt{[n]}a|n-1\rangle \qquad a^+|n\rangle = \sqrt{[n+1]}a|n+1\rangle. \tag{9}$$

3. Deformed fermionic algebras

Let us consider a structure function $F(x)$ which is a positive analytic function defined on the interval $[0, 2]$, such that

$$F(0) = 0 \qquad F(1) = 1 \qquad F(2) = 0. \tag{10}$$

Table 2. Structure functions for the bosonic representation of the fermionic algebra.

	$F(x)$	Reference
i	$\sin^2(\pi x/2)$	[36,37]
ii	$q^{x-1}\sin^2(\pi x/2)$	[11–16]
iii	$x(2-x)$	[26]
iv	$\dfrac{\sin(\tau x)\sin(\tau(2-x))}{\sin^2(\tau)}$	[27,28]

Examples of such functions are given in table 2.

The basis of the representation is degenerated to the states $|0\rangle$ and $|1\rangle$, because, by definition,

$$a^+|0\rangle = F(1)|1\rangle = |1\rangle \qquad a^+|1\rangle = 0 \tag{11}$$

(since $F(2) = 0$), and

$$(a^+)^2|0\rangle = 0 \qquad (a^+)^2|1\rangle = 0. \tag{12}$$

In the same way we can prove that

$$(a)^2 = (a^+)^2 = 0. \tag{13}$$

As an example we consider the case of the q-deformed fermions, which satisfy the q-deformed equation [11–16],

$$aa^+ + qa^+a = q^N. \tag{14}$$

The algebra of the operators $\{a, a^+, N\}$ satisfying equation (14) is given by

$$[a, N] = a \qquad [a^+, N] = -a^+ \tag{15}$$

and

$$a^+a = q^{N-1}\sin^2(N\pi/2) \qquad aa^+ = q^N\cos^2(N\pi/2) \tag{16}$$

corresponding to the structure function (ii) of table 2. It is clear that, in this case, equation (13) is satisfied.

Another example is provided by the usual fermions. Jannussis *et al* [36, 37] have introduced the notion of the bosonization of the fermions using a boson mapping of the fermionic algebra:

$$a = \frac{\cos(\pi n/2)}{\sqrt{n-1}}b \qquad a^+ = b^+\frac{\cos(\pi n/2)}{\sqrt{n-1}} \tag{17}$$

where $[b, b^+] = 1$, $b^+b = n$ is a bosonic algebra. This method is equivalent to the choice of the structure function (i) in table 2.

Yet another example is provided by the parafermionic oscillators introduced by Ohnuki and Kamefuchi [26], which can be described by the structure function:

$$F(x) = x(p-x) \qquad p = 2, 3, \ldots. \tag{18}$$

In the special case $p = 2$ this parafermionic oscillator has the characteristic fermionic behaviour described by equation (13). It corresponds to the structure function (iii) in table 2. It is worth noticing that the deformed oscillator described by the structure function of equation (18) corresponds to the deformed oscillator having the energy spectrum of a Pöschl–Teller potential [31] or the energy spectrum of a shifted Morse potential [39].

A further example is provided by the q-deformed parafermionic oscillator in [27,28], which corresponds to the structure function (iv) of table 2.

In all these algebras the number operator N has two possible eigenvalues, 0 and 1, therefore satisfying the relations:

$$N^2 = N \qquad (1-N)^2 = 1-N. \tag{19}$$

In all cases of deformed fermionic algebras in table 2, starting from equation (19) we can prove that

$$a^+a = N \qquad aa^+ = 1-N. \tag{20}$$

Therefore the known fermionic deformations of table 2 satisfy the usual fermionic algebra:

$$aa^+ + a^+a = 1 \qquad a^2 = (a^+)^2 = 0. \tag{21}$$

The inverse is also true: the usual fermionic algebra defined by equation (21) satisfies the properties (19) and (20), which implies that deformed relations such as the one in equation (14) are valid. In order to see this, one can write (with $q = \mathrm{e}^\tau$)

$$q^N = \mathrm{e}^{\tau N} = 1 + \tau N + \frac{\tau^2 N^2}{2!} + \frac{\tau^3 N^3}{3!} + \cdots.$$

Taking into account that in the case of fermions $N^2 = N$, this relation can be written as

$$\begin{aligned} q^N &= 1 + N\tau + N\frac{\tau^2}{2!} + N\frac{\tau^3}{3!} + \cdots \\ &= 1 + N(\mathrm{e}^\tau - 1) = 1 + (q-1)N = (1-N) + qN \end{aligned}$$

which coincides with equation (14) if equation (20) is taken into account.

These assertions lead to the conclusion that all the deformed fermionic algebras are isomorphically equivalent structures possessing the usual matrix representation:

$$a \longrightarrow \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a^+ \longrightarrow \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{22}$$

and

$$N \longrightarrow \frac{1+\sigma_0}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{23}$$

(One can easily verify that the matrices of equations (22)–(23) satisfy equations (19)–(21). Thus, because of the proof given after equation (21), they also satisfy equation (14).)

It should be noticed that the equivalence between the q-deformed fermions of equation (14) and the usual fermions has been proven in [40] by means of a transformation. Here this equivalence is proven for all versions of deformed fermions, not only for the particular deformation of equation (14).

4. Polynomial realization of the deformed fermionic algebras

For each deformed algebra a polynomial realization can be found [31,32]. This realization is the corresponding deformed version of the Bargmann representation [41] for the harmonic oscillator. These representations have the same structure as the usual Bargmann case but the notion of the derivative and integral should be deformed. The same is true for the q-deformed harmonic oscillator [5,6] (with real q). As a result, the q-analysis has been developed recently by many authors [42–48].

In [31] the deformed algebra corresponding to the Pöschl–Teller spectrum was studied, which is exactly the case of a deformed oscillator with a finite-dimensional Fock basis, while in [32] the deformed oscillator algebra corresponding to the Coulomb potential was studied, which is an example of a deformed oscillator algebra with an infinite-dimensional Fock basis but with a spectrum having one accumulation point. The usual oscillator algebra is an example of an oscillator with an infinite-dimensional basis and a spectrum without any accumulation points. The same is true for a q-deformed oscillator with real q, while the case with q equal to a root of unity is an example of a deformed algebra with a finite-dimensional Fock space.

All the deformed fermions defined in the previous section correspond to deformed oscillators with a two-dimensional Fock basis. In this section following the same method as in [31,32] we define the polynomial basis (Bargmann representation) of the fermionic algebra.

The polynomial realization of the deformed oscillator is defined as follows. Let $\mathcal{H}$ be the set of the entire functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n . \tag{24}$$

The projection operator J_k projects the function $f(z)$ to the truncated polynomial $J_k f(z)$ of degree k:

$$J_k f(z) = \sum_{n=0}^{k} a_n z^n \in J_k \mathcal{H}. \tag{25}$$

The space spanned by the deformed oscillator basis $\{|n\rangle, n = 0, 1, 2, \ldots, p-1\}$ is equivalent to the space $J_{p-1}\mathcal{H}$ spanned by the basis:

$$\frac{z^n}{\sqrt{[n]!}} \qquad n = 0, 1, \ldots, p-1. \tag{26}$$

The case of the finite-dimensional Fock space can be found in [31], while the case of an infinite-dimensional space can be found in [32].

In the deformed fermionic case the Fock space has a two-dimensional basis. Therefore, all the polynomials or functions involved should be truncated, conserving only the constant and the linear terms. The Fock basis corresponds to

$$|0\rangle \longrightarrow 1 \qquad |1\rangle \longrightarrow z \tag{27}$$

because in all cases in table 2 the structure function is normalized to unity:

$$F(1) = 1. \tag{28}$$

Any function $f(z) \in J_1\mathcal{H}$ can be written as follows

$$f(z) = \sum_{n=0}^{1} f_n \frac{z^n}{[n]!} = f_0 + f_1 z \equiv \langle z|f\rangle \tag{29}$$

where

$$\langle z| \equiv \sum_{n=0}^{1} \frac{z^n}{\sqrt{[n]!}}\langle n| = \langle 0| + z\langle 1| \tag{30}$$

and

$$|f\rangle \equiv \sum_{n=0}^{1} \frac{f_n}{\sqrt{[n]!}}|n\rangle = f_0|0\rangle + f_1|1\rangle. \tag{31}$$

The element $|z\rangle$ is the *coherent* (but not normalized) eigenstate of the destruction operator a with eigenvalue $\bar{z}$:

$$a|z\rangle = \bar{z}|z\rangle. \tag{32}$$

The multiplication of the function $f(z)$ by z can be regarded as an application from the space of entire functions $\mathcal{H}$ into $\mathcal{H}$. The restriction of this application in the space $J_1\mathcal{H}$ is formally represented by $J_1 z J_1$. This operation corresponds to the following one:

$$J_1\left(z \sum_{n=0}^{1} a_n z^n\right) = a_o z \in J_1\mathcal{H} - J_0\mathcal{H}. \tag{33}$$

The derivative $\partial/\partial z$ is also an application defined in the space $\mathcal{H}$. The restriction of this application in $J_1\mathcal{H}$ is easily calculated:

$$(\partial/\partial z) \sum_{n=0}^{1} a_n z^n = a_1 \in J_0\mathcal{H}. \tag{34}$$

In the space $J_1\mathcal{H}$ we can define the operator

$$\frac{\partial}{\partial_D z} \boxminus \frac{1}{z} F\left(z \circ \frac{\partial}{\partial z}\right). \tag{35}$$

Without difficulty we can show that

$$\frac{\partial}{\partial_D z} z^n = F(n) z^{n-1} \qquad \text{if } n = 0, 1. \tag{36}$$

The normalization of the fermionic structure functions (equation (28)) implies that the deformed derivation coincides with the usual derivation in all fermionic cases of table 2. Thus we have

$$\frac{\partial}{\partial_D z} f(z) = \left(J_1 \circ \frac{\partial}{\partial z} \circ J_1\right) f(z). \tag{37}$$

The algebra of the creation and destruction operators corresponds to

$$a \longrightarrow J_1 \circ z \circ J_1 \tag{38}$$

$$a^+ \longrightarrow \frac{\partial}{\partial_D z} = \left(J_1 \circ \frac{\partial}{\partial z} \circ J_1\right) \tag{39}$$

$$N \longrightarrow z \circ \partial/\partial z. \tag{40}$$

Using the above correspondences the familiar fermion properties can be verified:

$$\left(aa^+\right)|f\rangle \longrightarrow (J_1 \circ z \circ J_1) \circ \left(J_1 \circ \frac{\partial}{\partial z} \circ J_1\right)(f_0 + f_1 z) = f_1 z \tag{41}$$

$$\left(a^+ a\right)|f\rangle \longrightarrow \left(J_1 \circ \frac{\partial}{\partial z} \circ J_1\right) \circ (J_1 \circ z \circ J_1)(f_0 + f_1 z) = f_0 \tag{42}$$

therefore

$$\left\{a, a^+\right\}|f\rangle = |f\rangle. \tag{43}$$

The operator $z \circ \partial/\partial_D z$ defined by equation (35) is a one-to-one application in the subspace $J_1\mathcal{H} - J_0\mathcal{H}$. Therefore the inverse of this operator exists in this subspace and it is given by

$$\left(z \circ \frac{\partial}{\partial_D z}\right)^{-1} z \equiv \frac{1}{F(1)} z. \tag{44}$$

Using the above operator the integration operator Int_D can be defined by

$$\mathrm{Int}_D \equiv \left(z \circ \frac{\partial}{\partial_D z}\right)^{-1} \circ z. \tag{45}$$

Without difficulty the following relation can be shown:

$$\mathrm{Int}_D z^0 \equiv \frac{z^1}{F(1)} = z \qquad \mathrm{Int}_D z = 0. \tag{46}$$

For any function $f(z)$, given by equation (29), we can define the integral

$$\mathrm{Int}_D f(z) \equiv \int_0^z f(u)\, \mathrm{d}_D u = f_0 z \tag{47}$$

and, by definition,

$$\int_a^b f(u)\, \mathrm{d}_D u = \mathrm{Int}_D f(b) - \mathrm{Int}_D f(a). \tag{48}$$

The definition of the coherent state of equation (32) implies that the deformed exponential function is defined by

$$\exp_D(z\bar{w}) \equiv \langle z|w\rangle = \sum_{n=0}^{1} \frac{(z\bar{w})^n}{[n]!} = 1 + z. \tag{49}$$

Without difficulty we can show the deformed generalizations of the usual identities:

$$\int_0^z \left(\frac{\partial}{\partial_{\mathrm{D}} u}\right) f(u)\,\mathrm{d}_{\mathrm{D}} u = f(z) - f(0) \tag{50}$$

$$\left(\frac{\partial}{\partial_{\mathrm{D}} z}\right) \int_0^z f(u)\,\mathrm{d}_{\mathrm{D}} u = f(z) - f'(0) \tag{51}$$

$$\left(\frac{\partial}{\partial_{\mathrm{D}} w}\right) \exp_{\mathrm{D}}[wz] = J_1 z \circ \exp_{\mathrm{D}}[wz] = z \tag{52}$$

$$\int_0^z \exp_{\mathrm{D}}[wu]\,\mathrm{d}_{\mathrm{D}} u = w^{-1}(\exp_{\mathrm{D}}[wz] - 1) = z. \tag{53}$$

The space $J_1\mathcal{H}$ also has the structure of a two-dimensional Hilbert space with the product

$$\langle f|g\rangle \equiv \left[\bar{f}\left(\frac{\partial}{\partial_{\mathrm{D}} z}\right) g(z)\right]_{z=0}. \tag{54}$$

The above defined product can be formally generated by introducing a measure $\mathrm{d}\mu\,(\bar{z}, z)$ on the complex z plane having the property

$$\int \mathrm{d}\mu\,(\bar{z}, z)\bar{z}^n z^m \equiv \delta_{n,m}[n]! \tag{55}$$

and

$$\langle f|g\rangle = \int \mathrm{d}\mu\,(\bar{z}, z)\bar{f}(\bar{z})g(z). \tag{56}$$

If A is an operator defined on the two-dimensional Hilbert space spanned by the vectors $|0\rangle, |1\rangle$, then there is a matrix representation defined by

$$A = \sum_{m,n=0}^{1} A_{n,m}|n\rangle\langle m|. \tag{57}$$

This operator corresponds to a kernel $A(w, u)$ acting on the space $J_1\mathcal{H}$:

$$A(w, \bar{u}) \equiv \langle w|A|u\rangle = \sum_{m,n=0}^{p-1} A_{n,m}\langle w|n\rangle\langle m|u\rangle = \sum_{m,n=0}^{1} A_{n,m}\frac{w^n}{\sqrt{[n]!}}\frac{\bar{u}^m}{\sqrt{[m]!}} \tag{58}$$

and

$$\langle w|A|f\rangle = \int \mathrm{d}\mu\,(\bar{z}, z)A(w, \bar{z})f(z) \tag{59}$$

where $f(z) = f_0 + f_1 z$.

The product of two operators A and B corresponds to the *convolution* of the corresponding kernels:

$$\langle w|AB|u\rangle = \int \mathrm{d}\mu(\bar{z}, z) A(w, \bar{z}) B(z, \bar{u}). \tag{60}$$

This equation implies the following *resolution* of the identity:

$$\int \mathrm{d}\mu(\bar{z}, z)|z\rangle\langle z| = \sum_{n=0}^{p-1} |n\rangle\langle n| \equiv \mathbf{1} \tag{61}$$

where $\mathbf{1}$ is the unity in the two-dimensional Hilbert space spanned by the vector basis $|n\rangle$, $n = 0, 1$. The following relations can be proven without difficulty, using the definition (equation (49)) of the deformed exponential function:

$$\int \mathrm{d}\mu(\bar{z}, z) \exp_{\mathrm{D}}(u\bar{z}) f(z) = f(u) \tag{62}$$

$$\int \mathrm{d}\mu(\bar{z}, z) \exp_{\mathrm{D}}(u\bar{z}) \exp_{\mathrm{D}}(z\bar{w}) = \exp_{\mathrm{D}}(u\bar{w}) \tag{63}$$

$$\int \mathrm{d}\mu(\bar{z}, z) \exp_{\mathrm{D}}(z\bar{z}t) = \sum_{n=0}^{1} t^n = 1 + t. \tag{64}$$

If A is an operator defined by equation (59), then

$$\int \mathrm{d}\mu(\bar{z}, z) A(z, \bar{z}) = \mathrm{Tr}(A) = \sum_{n=0}^{1} A_{n,n}. \tag{65}$$

These formulae indicate that the measure $\mathrm{d}\mu(\bar{z}, z)$ has the basic properties of a Gaussian measure.

5. Discussion

Several deformed fermionic algebras (summarized in table 2) have been considered. It has been proven that all of them are mutually equivalent, as well as being equivalent to the usual fermionic algebra, if the fermionic annihilation and creation operators obey the relations:

$$a^2 = (a^+)^2 = 0 \tag{66}$$

because they accept the same polynomial realization. As a result, only one type of fermion with the property of equation (66) can exist (the usual fermion), while in the case of bosons several different versions of deformed bosons can exist (some of them are shown in table 1). (It is worth remembering at this point that the physical properties implied by different versions of deformed bosons can be significantly different (see [33,34] for an example).)

The uniqueness of the fermionic algebra proven in this paper simplifies considerably the problem of representing q-deformed superalgebras (see [17–20] for some examples) in a coupled basis of bosons and fermions. Work in this direction is in progress.

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